

# ENUMERATIVE GEOMETRY OF HYPERELLIPTIC PLANE CURVES

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## 0. INTRODUCTION

In recent years there has been a tremendous amount of progress on classical problems in enumerative geometry. This has largely been a result of new ideas and motivation for these problems coming from theoretical physics. In particular, the theory of Gromov-Witten invariants has provided powerful tools for counting curves satisfying incidence conditions.

This theory has been most successful in dealing with questions about rational curves. This is partly because it is much more common for the genus 0 invariants to correspond to enumerative problems. In addition, it is much easier to compute these invariants. This is due mainly to the existence of the WDVV equations. There has been success in extending the techniques used to derive these equations to find recursions satisfied by the invariants in genus 1 and 2. ([G],[BP]). In some situations these higher genus invariants also correspond to classical enumerative problems. Thus, the theory gives new methods to solve these problems.

We will explore a different approach to using Gromov-Witten theory to solve enumerative problems involving higher genus curves. Rather than generalizing the methods that succeed in genus 0, we try to reduce questions in higher genus to questions about rational curves. We utilize the well-developed theory of genus 0 Gromov-Witten invariants to solve enumerative problems involving hyperelliptic curves in  $\mathbf{P}^2$ . Our main enumerative result is the construction of a recursive algorithm which counts the number of hyperelliptic plane curves of degree  $d$  and genus  $g$  passing through  $3d + 1$  general points.

The basic facts about stable maps and Gromov-Witten theory are reviewed in Section 1. The main results we need are the WDVV equations which allow us to compute, and an explicit representation of the virtual fundamental class as a Chern class.

In Section 2 we introduce the main idea of the paper. Thinking of a map from a hyperelliptic curve as a family of maps from pairs of points parametrized by  $\mathbf{P}^1$  gives us a natural correspondence between

hyperelliptic plane curves and rational curves in  $H = H(2, \mathbf{P}^2)$ , the Hilbert scheme of two points in the plane. Furthermore, the condition that the hyperelliptic curve meet a point  $p$  in  $\mathbf{P}^2$  is equivalent to the condition that the associated rational curve in  $H$  meets the cycle

$$\Gamma(p) = \{\text{subschemas incident to } p\}.$$

Since genus 0 Gromov-Witten invariants naively correspond to counting rational curves incident to cycles, it is reasonable to hope that we could understand our enumerative problem in terms of Gromov-Witten theory on  $H$ . We study the geometry of this Hilbert scheme, and identify which curve classes arise via this correspondence.

These results enable us to identify exactly how the Gromov-Witten invariants differ from the solution to the enumerative problem in which we are interested. There are extraneous components of the space of maps which contribute to the invariants. In order to relate the Gromov-Witten theory to enumerative geometry, it is necessary to identify the contributions from these components. This is carried out in section 3. By studying the deformation theory on these components, we can relate their contributions to a much simpler problem about curves on a blow up of the plane where it is trivial to determine the solution. The conclusion is a simple formula relating the enumerative numbers to the Gromov-Witten invariants. If we set  $E(d, g)$  to be the number of degree  $d$  genus  $g$  hyperelliptic curves passing through  $3d + 1$  general points in the plane, and set  $I(d, g)$  to be the corresponding Gromov-Witten invariant, then our main result is that

$$I(d, g) = \sum_{h \geq g} \binom{2h+2}{h-g} E(d, h).$$

This relationship can be inverted to solve the enumerative problem in terms of Gromov-Witten invariants. We actually get a similar formula for the more exotic enumerative problem of counting hyperelliptic plane curves passing through fixed points, with certain pairs of the points required to be hyperelliptically conjugate. This condition can be used to recover genus 0 and 1 Severi degrees from this formalism. We also use our calculations to recover a formula of Abramovich and Bertram [AB] about genus 0 Severi degrees on the Hirzebruch surfaces  $\mathbb{F}_0$  and  $\mathbb{F}_2$ .

In Section 4 we show how we can use the formal properties of the invariants to give a recursive algorithm computing the  $I(d, g)$ . In this way we are able to effectively compute the solution to our original problem. In addition, we use our calculations to give an explicit presentation of the small quantum cohomology ring of  $H$ .

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## 1. GROMOV-WITTEN INVARIANTS

**1.1. Stable Maps.** The Gromov-Witten invariants of a smooth projective variety  $X$  are defined as integrals over the space of stable maps,  $\overline{M}_{g,n}(X, \beta)$ . We refer the reader to [FP] for a construction of this space, and a careful discussion of its properties. We should note that while in that reference the emphasis is on the coarse moduli scheme, we will take all our moduli spaces to be in the category of Deligne-Mumford stacks. We will briefly review some of the fundamental structures on this space to fix our notation.

We denote by  $M_{g,n}(X, \beta)$  the open substack of  $\overline{M}_{g,n}(X, \beta)$  which parametrizes maps from smooth irreducible curves. While we think of  $\overline{M}_{g,n}(X, \beta)$  as a compactification of this space, it is entirely possible for a component of the compact space to parametrize only maps from singular curves. In other words,  $M_{g,n}(X, \beta)$  need not be dense in  $\overline{M}_{g,n}(X, \beta)$ .

These spaces come equipped with several natural morphisms. For each of the  $n$  marked points, there is an evaluation map,

$$\rho_i : \overline{M}_{g,n}(X, \beta) \rightarrow X,$$

which takes the moduli point  $(C; x_1, \dots, x_n; f)$  to the point  $f(x_i)$ . There is the map

$$\pi_{n+1} : \overline{M}_{g,n+1}(X, \beta) \rightarrow \overline{M}_{g,n}(X, \beta)$$

which simply forgets the last marked point (and stabilizes the curve if necessary.) This realizes  $\overline{M}_{g,n+1}(X, \beta)$  as the universal curve over  $\overline{M}_{g,n}(X, \beta)$ . The universal morphism is given by the  $(n+1)$ st evaluation

map.

$$\begin{array}{ccc} \overline{M}_{g,n+1}(X, \beta) & \xrightarrow{\rho_{n+1}} & X \\ \pi_{n+1} \downarrow & & \\ \overline{M}_{g,n}(X, \beta) & & \end{array}$$

In fact, given any subset of  $\{1, \dots, n\}$  there is the analogous map which forgets all marked points in that subset. We will only have use for this construction when the subset is all of the points, and we will denote the corresponding map from  $\overline{M}_{g,n}(X, \beta)$  to  $\overline{M}_{g,0}(X, \beta)$  by  $\pi$ . Lastly, there is a natural morphism

$$\eta : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}$$

which simply forgets the map (and again stabilizes the curve if necessary.)

**1.2. Deformation Theory.** Crucial to the Gromov-Witten theory is a good understanding of the local structure of these moduli spaces. There is a natural obstruction theory for stable maps which formally locally realizes the moduli space as the zero locus of a collection of equations in its Zariski tangent space.

At a fixed moduli point, this theory consists of two vector spaces, a tangent space and an obstruction space. These spaces are  $\text{Ext}^1(f^*\Omega_X \rightarrow \Omega_C, \mathcal{O}_C)$  and  $\text{Ext}^2(f^*\Omega_X \rightarrow \Omega_C, \mathcal{O}_C)$  respectively. These varying vector spaces can also be thought of as global coherent sheaves on the moduli space. From here on, these two sheaves will be denoted by  $TM$  and  $\mathcal{E}$ .

There are two facts about these sheaves that will be important for us. The first is that on the smooth locus of the moduli space, both are locally free. For  $TM$  this is true by definition, and it follows that  $\mathcal{E}$  is locally free from the existence of a presentation for these two sheaves as a kernel and cokernel of a two term complex of vector bundles.

The other thing we need to know about them is that they fit into the following exact sequence:

$$(1.1) \quad \begin{aligned} 0 \rightarrow \text{Ext}^0(\Omega_C, \mathcal{O}_C) &\rightarrow H^0(f^*TX) \rightarrow TM \rightarrow \\ &\rightarrow \text{Ext}^1(\Omega_C, \mathcal{O}_C) \rightarrow H^1(f^*TX) \rightarrow \mathcal{E} \rightarrow 0. \end{aligned}$$

(Here, and throughout the paper, we will often refer to sheaves by simply naming their fibers. So the  $H^i$  should be thought of as  $R^i\pi_*$  and similarly for the Ext groups.)

To understand the geometry of this sequence, it should be observed that the  $\text{Ext}^i(\Omega_C, \mathcal{O}_C)$  are the spaces of automorphisms and deformations of the underlying nodal curve, and the  $H^i(C, f^*TX)$  are the

tangent and obstruction spaces to the space of maps from the fixed curve  $C$  to  $X$ .

Notice that if  $H^1(f^*TX)$  vanishes, we can conclude that  $\mathcal{E}$  vanishes also. This forces the moduli space to be smooth of the expected dimension. We will also want to know the following stronger fact, which is an easy consequence of Theorem II.1.7 in [K].

**Theorem 1.2.** *If  $f : C \rightarrow X$  is a morphism from a stable pointed curve such that  $H^1(C, f^*TX) = 0$ , the forgetful morphism*

$$\eta : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}$$

*is smooth at  $[f]$ .*

**1.3. Virtual Fundamental Class.** The main technical point in defining the Gromov-Witten invariants is the construction of a natural Chow homology class in  $A_*(\overline{M}_{g,n}(X, \beta))$  – the *virtual fundamental class* which we denote by  $[\overline{M}_{g,n}(X, \beta)]^{vir}$ . It is pure dimensional of the expected dimension

$$d_X(\beta) = -K_X \cdot \beta + (\dim(X) - 3)(1 - g) + n.$$

Given this class, the Gromov-Witten invariants are defined to be multilinear maps from the cohomology of  $X$  to  $\mathbb{Q}$  given by

$$I_{g,\beta}^X(\gamma_1 \cdots \gamma_n) = \int_{[\overline{M}_{g,n}(X, \beta)]^{vir}} \rho_1^* \gamma_1 \cup \cdots \cup \rho_n^* \gamma_n.$$

This virtual class is constructed in [LT] and in [BF], [B]. In general, the construction is quite subtle. Fortunately, we will need to know relatively little. One fact we need is that the virtual class on the pointed map space is pulled back from the unpointed space by the flat morphism  $\pi$  which forgets the marked points. Since it is generally easier to think about these unpointed spaces anyway, we will do all of our calculations there.

On the unpointed spaces, we will need to compute only the restriction of the virtual class to the smooth locus of the moduli space. This can be realized as the top Chern class of the obstruction bundle  $\mathcal{E}$ . Essentially this is a manifestation of the standard fact that for smooth varieties, excess intersection classes can be described as Chern classes of associated vector bundles.

In particular, the virtual class can naturally be thought of as a cohomology class in this situation. We will tend to think of the  $\gamma_i$  as homology classes, effectively reversing the usual formulation. That is, we will represent the cohomology classes  $\gamma_i$  by algebraic cycles  $\Gamma_i$ , pull these back to the moduli space, intersect them, and integrate the virtual class over the resulting cycle. As we want to work on the unpointed

space, we will actually push this cycle forward to  $\overline{M}_{g,0}(X, \beta)$  and integrate the appropriate Chern class over the image (up to a possible multiplicity.) The formulation we ultimately want is the following.

**Theorem 1.3.** *Suppose  $\Gamma_1, \dots, \Gamma_n$  are cycles in  $X$  representing the cohomology classes  $\gamma_i$  such that  $\rho_i^{-1}(\Gamma_i)$  intersect generically transversally. Then if*

$$\pi_*([\cap_i \rho_i^{-1}(\Gamma_i)]) = A$$

*where  $A$  is a cycle contained in the smooth locus of  $\overline{M}_{g,0}(X, \beta)$ ,*

$$I_{g,\beta}^X(\gamma_1, \dots, \gamma_n) = \int_A c_{\text{top}}(\mathcal{E}).$$

The only real content of this statement is the previously mentioned identification of the virtual class in this setting with the top Chern class of the obstruction bundle. This is Proposition 5.6 in [BF].

**1.4. Properties of Gromov-Witten Invariants.** The Gromov-Witten invariants satisfy many formal properties which can make computing them more tractable than solving enumerative problems in general.

First of all, the motivating property satisfied by the invariants is that they are invariant under deformations of  $X$ . There are several ways to express this property, but essentially the point is that since smooth families of varieties are locally trivial in the  $C^\infty$  category, there are canonical isomorphisms between the cohomology groups of nearby fibers. With respect to this identification, the Gromov-Witten invariants are independent of the choice of fiber.

Another formal property of the invariants that we will use is the so called *divisor axiom*

$$I_{g,\beta}^X(\gamma_1 \cdots \gamma_n) = (\gamma_1 \cap \beta) \cdot I_{g,\beta}^X(\gamma_2 \cdots \gamma_n).$$

Repeated use of this axiom reduces the computation of any Gromov-Witten invariant to one which involves no divisor classes.

The genus 0 invariants behave better in several ways than the higher genus invariants. Since we will work with only the genus 0 Gromov-Witten invariants, from now on we use  $I_\beta^X(\gamma_1 \cdots \gamma_n)$  to mean  $I_{0,\beta}^X(\gamma_1 \cdots \gamma_n)$ . We will also suppress the  $X$  from the notation except where there is possibility of confusion.

The invariants can be computed much more readily in the genus 0 case than in general. There are essentially two reasons for this. One is that it is much more common for the  $H^1(f^*TX)$  to vanish for genus 0 maps which means that for sufficiently nice varieties, there are no virtual class considerations. This is the case, for example, for all homogeneous varieties.

The other more general fact is that the genus 0 invariants satisfy an important family of recursive equations which often determine all of the invariants from a small amount of initial data. These relations arise by considering the many natural morphisms from the moduli space to  $\mathbf{P}^1$  given by composing the map  $\eta : \overline{M}_{0,n}(X, \beta) \rightarrow \overline{M}_{0,n}$  with a map from  $\overline{M}_{0,n}$  to  $\overline{M}_{0,4}$  given by forgetting all but 4 of the points. As  $\overline{M}_{0,4} \cong \mathbf{P}^1$ , different points on it are rationally equivalent. By pulling back the points corresponding to reducible curves, one obtains linear equivalences between divisors on  $\overline{M}_{0,n}(X, \beta)$ . The WDVV equations are then deduced by intersecting these divisors with curves to obtain numerical equalities. These arguments all ultimately rest on nice properties of the virtual fundamental classes. We refer the reader to [KM] or [FP] for a careful discussion of this construction in the homogeneous case, and to [B], [LT] for the additional arguments needed to handle the virtual considerations needed in general.

The final result can be described as follows. If we choose a homogeneous basis for the cohomology of  $X$  given by  $T_0, T_1, \dots, T_m$ , then we get a relation for each diagram of the following form

$$\left( \begin{smallmatrix} i & l \\ j & k \end{smallmatrix} \right) \sim \left( \begin{smallmatrix} i & l \\ j & k \end{smallmatrix} \right)$$

Typically the relation is described by first constructing a generating function using the Gromov-Witten invariants as coefficients, and then producing a differential equation which is satisfied by that function. This formalism is in general very useful, but for our purposes we really just want to know the explicit recursive equations that arise from identifying coefficients in this expression. To this end, we need to choose a collection of cohomology classes on  $X$ ,  $\gamma_1, \dots, \gamma_n$ , and a class  $\beta$  in  $H_2(X, \mathbb{Z})$ . The relation that is obtained is the following:

$$(1.4) \quad \begin{aligned} & \sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = [n] \\ e, f}} I_{\beta_1}(T_i \cdot T_j \cdot T_e \cdot \prod_{a \in A} \gamma_a) g^{ef} I_{\beta_2}(T_k \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b) \\ &= \sum_{\substack{\beta_1 + \beta_2 = \beta \\ A \cup B = [n] \\ e, f}} I_{\beta_1}(T_i \cdot T_l \cdot T_e \cdot \prod_{a \in A} \gamma_a) g^{ef} I_{\beta_2}(T_j \cdot T_k \cdot T_f \cdot \prod_{b \in B} \gamma_b). \end{aligned}$$

Because the space of degree 0 maps is so simple, the terms in the summation with  $\beta_1$  or  $\beta_2$  equal to 0 have a particularly simple form. Summing over all the remaining indices, the contribution of these terms

on the left hand side of 1.4 is just

$$I_\beta(T_i \cdot T_j \cdot T_k \cup T_l \cdot \prod_1^n \gamma_i) + I_\beta(T_k \cdot T_l \cdot T_i \cup T_j \cdot \prod_1^n \gamma_i).$$

Of course, the analogous expression gives the  $\beta = 0$  terms for the right hand side of 1.4 as well. Using these equations, as well as the divisor axiom, Kontsevich and Manin show in [KM] that on a variety whose cohomology is generated by divisors, all the genus zero Gromov-Witten invariants can be recursively determined by knowledge of the two point invariants,  $I_\beta(\gamma_1 \cdot \gamma_2)$ .

## 2. THE HILBERT SCHEME

**2.1. Geometry of the Hilbert Scheme.** The variety whose Gromov-Witten invariants we will be interested in here is  $H = H(2, \mathbf{P}^2)$ , the Hilbert scheme of two points in the plane. It functorially parametrizes length two subschemes of  $\mathbf{P}^2$ . So there exists a subscheme  $\mathcal{Z}$  in  $H \times \mathbf{P}^2$  whose fiber over a point of  $H$  is exactly the subscheme parametrized by that point.

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{f} & \mathbf{P}^2 \\ \nu \downarrow & & \\ H & & \end{array}$$

In this section, we collect some basic facts about the geometry of  $H$ .

Given a degree two subscheme,  $S \subset \mathbf{P}^2$ , there exists a unique line containing  $S$ . This correspondence gives rise to a morphism

$$\pi : H \rightarrow \mathbf{P}^{2*}.$$

The fiber over a point  $[L]$  is the set of subschemes contained in that line. This is canonically  $\text{Sym}^2(L)$ . So we can realize  $H$  as a  $\mathbf{P}^2$  bundle over  $\mathbf{P}^{2*}$ . It is simply  $\mathbf{P}(\text{Sym}^2(S))$  where  $S$  is the tautological rank 2 subbundle on  $\mathbf{P}^{2*}$ .

One thing that we can see immediately from this description of  $H$  is that it is a smooth variety. It follows that  $\mathcal{Z}$  is also smooth, and the morphism from  $\mathcal{Z}$  to  $H$  is finite, and simply branched over  $\Delta$ , the smooth subvariety of  $H$  parametrizing nonreduced length two subschemes.

This representation of  $H$  as a projective bundle also gives us a very good understanding of its cohomology ring. First of all, we can conclude that the Chow ring and cohomology ring are isomorphic, so we will use Chow notation from here on to avoid doubling indices. We see that  $\text{Pic}(H)$  has rank 2 and is generated by classes  $T_1 = \pi^*(\text{hyperplane})$

and  $T_2 = \mathcal{O}(1)$ .  $T_1$  is the divisor consisting of schemes whose associated line is incident to a fixed point.  $T_2$  can be represented by the divisor consisting of all subschemes incident to a fixed line in  $\mathbf{P}^2$ .

From these representations, it is clear that both of the classes  $T_1$  and  $T_2$  move without basepoints. It follows that any effective curve in  $H$  has non-negative intersection with each of them. In fact, the converse is true. If we set  $B_1$  to be the curve of subschemes supported at a fixed point, and  $B_2$  to be a line in a fiber of  $\pi$ , we see that

$$B_i \cdot T_j = \delta_{ij}.$$

This implies that the cone of effective curves consists exactly of positive integral combinations of  $B_1$  and  $B_2$ , so we will fix these as our basis for  $A_1 H$  and use  $(a, b)$  to denote  $aB_1 + bB_2$  throughout.

We know that  $A^*(H)$  is generated by  $T_1$  and  $T_2$  modulo the relations  $T_1^3 = 0$  and

$$T_2^3 + c_1 T_2^2 + c_2 T_2 = 0$$

where  $c_i = \pi^* c_i(\text{Sym}^2(S))$ . (Note that  $c_3$  must vanish for dimension reasons.) We can easily compute these Chern classes. They are

$$\begin{aligned} c_1 &= -3T_1 \\ c_2 &= 6T_1^2. \end{aligned}$$

If we define  $T_3 = T_1^2$  and  $T_4 = T_1 T_2 - 2T_1^2$  then we can extend these to a Poincare dual basis of the integral Chow ring of  $H$  by which we mean a basis satisfying

$$\int T_i \cup T_{8-j} = \delta_{ij}.$$

Here we take  $T_0$  to be the fundamental class and  $T_8$  to be the point class. Our choice of  $T_4$  is made because this is the class represented by the cycle

$$\Gamma(p) = \{\text{subschemas incident to } p\}$$

where  $p$  is a point of  $\mathbf{P}^2$ . Notice that it is clear that  $\int T_4 \cup T_4 = 1$  and  $\int T_4 \cup T_3 = 0$  as these correspond to the statements that there is a unique degree two subscheme of  $\mathbf{P}^2$  meeting two general points, and that there are no subschemes contained in a general line and meeting a general point. Also, we have already constructed  $T_6$  and  $T_7$  as our basis for the curve classes. The only remaining element of the basis is  $T_5 = T_1^2 - T_1 T_2 + T_2^2$  for which we know of no direct geometric interpretation. However, the class  $S_5 = T_5 + T_3$  is represented by the closure of the locus  $\{(p, q) : p \in l_1, q \in l_2\}$  where  $l_1$  and  $l_2$  are distinct lines in  $\mathbf{P}^2$ .

The description of  $H$  as a projective bundle also allows us to compute its canonical class. The result is that  $c_1(TH) = 3T_2$ .

We will see that it will be especially important for us to understand how the diagonal,  $\Delta$ , which parametrizes length 2 subschemes of  $\mathbf{P}^2$  supported at a single point, sits inside  $H$ . If we think of  $H$  as  $\mathbf{P}(\mathrm{Sym}^2 S)$ , then  $\Delta$  is the image of  $\mathbf{P}(S)$  under the degree 2 Veronese embedding. From this we can deduce that  $\mathrm{Pic}(\Delta)$  is generated by  $T_1$  and  $\frac{1}{2}T_2$ . Again, these are nef divisors so any curve in  $\Delta$  has positive integral intersection with each of them. Since  $B_1$  and  $2B_2$  can both be represented by curves in  $\Delta$  we see that the cone of effective curves in  $\Delta$  is just the locus  $(a, b)$  with  $a$  and  $b$  nonnegative, and  $b$  even. By intersecting it with curves, we can calculate that as a divisor in  $H$ ,

$$\Delta \equiv 2(T_2 - T_1).$$

**2.2. Geometry of Hyperelliptic Curves.** Although the method we use to solve enumerative problems is to immediately forget about the actual hyperelliptic curves and replace them with rational curves in  $H$ , we will discuss in this section the natural space of hyperelliptic maps, and make explicit the relationship between this space and the genus 0 space on which we calculate. In addition to clarifying our strategy, we will eventually make use of this direct construction to do some of the deformation theory calculations in Section 3.

The fundamental result about hyperelliptic curves which makes their geometry much easier to study than that of arbitrary curves is the following.

**Lemma 2.1.** *If  $f : C \rightarrow \mathbf{P}^r$  is a morphism from a hyperelliptic curve which does not factor through the hyperelliptic map, then  $f^*(\mathcal{O}(1))$  has no higher cohomology.*

*Proof.* In genus 0 or 1 all line bundles of positive degree have no higher cohomology, so the statement is vacuous. In genus 2 or higher, we know that the canonical morphism is 2 to 1 onto a rational normal curve. Given a divisor  $D$  in the linear series, Serre duality says that the dimension of  $H^1(D)$  is given by the dimension of  $H^0(K - D)$  which is sections of the canonical bundle vanishing on the divisor, or equivalently, hyperplanes in  $\mathbf{P}(H^0(K)^\vee)$  containing the image of the divisor under the canonical morphism. Since any collection of distinct points on a rational normal curve are in general linear position, the only way that a divisor can fail to impose the maximum number of conditions on the canonical series is for it to contain a pair of hyperelliptically conjugate points. But if every element of the linear series contains a

conjugate pair, we can deduce that the map must factor through the hyperelliptic map.  $\square$

We will make use of this fact to study one natural moduli space of hyperelliptic plane curves. This space, which we denote by  $H_g(\mathbf{P}^2, d)$  parametrizes hyperelliptic curves of genus  $g$  mapping to the plane. It is constructed as a locally closed subvariety of  $M_{g,0}(\mathbf{P}^2, d)$ .

Look at the locus  $H_g \subset M_g$  consisting of hyperelliptic curves. This locus is in fact a smooth substack of  $M_g$ . This follows from the standard realization of the moduli space of hyperelliptic curves as the quotient by the symmetric group of  $M_{0,2g+2}$ . In other words, a hyperelliptic curve is determined up to the hyperelliptic involution by the associated configuration of branch points on  $\mathbf{P}^1$ . We first define the space of hyperelliptic maps to be the preimage of this hyperelliptic locus in  $M_{g,0}(\mathbf{P}^2, d)$ . Call this space  $\tilde{H}_g(\mathbf{P}^2, d)$ .<sup>1</sup>

If we look at the open subset of this space which parametrizes maps which are birational onto their image, we get a reasonable candidate for a moduli space of hyperelliptic plane curves. It will be convenient for us to look at a slightly larger open subset. This space, which we will denote by  $H_g(\mathbf{P}^2, d)$ , is the subset parametrizing maps which do not factor through the hyperelliptic involution.

In this context, the significance of the vanishing result introduced earlier is the following.

**Theorem 2.2.** *The natural morphism from  $H_g(\mathbf{P}^2, d)$  to  $H_g$  is smooth.*

*Proof.* Consider the Euler sequence:

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus 3} \rightarrow T\mathbf{P}^2 \rightarrow 0.$$

It induces a surjection  $H^1(C, f^*\mathcal{O}(1))^{\oplus 3} \rightarrow H^1(C, f^*T\mathbf{P}^2)$ . We know by Lemma 2.1 that the first group vanishes, so we can conclude the smoothness of the morphism by Theorem 1.2.  $\square$

**Corollary 2.3.**  *$H_g(\mathbf{P}^2, d)$  is smooth and irreducible.*

*Proof.* The smoothness follows immediately, since  $H_g$  itself is smooth. Since  $H_g$  is irreducible, the corollary will follow once we verify that the fibers of  $\nu$  are irreducible. A fiber is just the set of all maps from a fixed

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<sup>1</sup>In genus 0 or 1, the space of hyperelliptic curves should really be thought of as a smooth family over the moduli space of curves.  $H_g$  should parametrize the choice of curve along with the choice of hyperelliptic involution. In this case, rather than a preimage, the hyperelliptic map space should be constructed as the fiber product of  $H_g$  with the map space. Throughout the paper, we will think of hyperelliptic curves in this way. That is, the term hyperelliptic curve should be taken to mean a curve together with a choice of hyperelliptic involution.

hyperelliptic curve to  $\mathbf{P}^2$  which do not factor through the hyperelliptic map. By associating to a map, the corresponding line bundle, we see that this space is fibered over  $\text{Pic}^d(C)$ . By Lemma 2.1 we can see that its image in  $\text{Pic}^d(C)$  is the open subset consisting of line bundles with vanishing first cohomology. Over this locus, the moduli space is just an open subset of the bundle of 3-tuples of sections modulo scalars.  $\square$

We will frequently make implicit use of the irreducibility statement in that it gives us a meaningful notion of a generic hyperelliptic plane curve.

**2.3. The Basic Correspondence.** Viewing the space in this way, one would be inclined to take as a compactification of this space the preimage of the closure of the hyperelliptic locus. However, this compactification differs markedly from the one we will study.

Instead, we make use of the following correspondence. Given a hyperelliptic map, we construct a map from a rational curve to  $H$ . The construction of this map is a simple application of the universal property of  $H$ . A hyperelliptic map gives us the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{g} & \mathbf{P}^2 \\ 2-1 \downarrow & & \\ \mathbf{P}^1 & & \end{array}$$

The graph of this correspondence gives a closed subscheme of  $\mathbf{P}^1 \times \mathbf{P}^2$  flat over  $\mathbf{P}^1$ . (Flatness is trivial here since  $\mathbf{P}^1$  is a smooth curve, and any surjective morphism from an irreducible variety to a smooth curve is flat.) Provided that  $g$  does not factor through the hyperelliptic map, the general fiber of the projection of the graph is a pair of distinct points. Thus the universal property of  $H$  gives us a natural morphism  $f : \mathbf{P}^1 \rightarrow H$ .

Conversely, if we have a map from  $\mathbf{P}^1$  to  $H$ , pulling back the universal subscheme gives us a scheme of degree 2 over  $\mathbf{P}^1$ , together with a map from this scheme to  $\mathbf{P}^2$ . It will not necessarily be the case that this scheme is what we would usually think of as a hyperelliptic curve. For example, if the map takes  $\mathbf{P}^1$  entirely into the diagonal, this scheme will be everywhere non-reduced. If we assume that the original morphism is transverse to  $\Delta$ , then this pull-back will be a smooth hyperelliptic curve. This is because  $\Delta$  is exactly the branch locus of the morphism from the universal subscheme to  $H$ .

The points of intersection of the rational curve in  $H$  with  $\Delta$  correspond exactly to the branch points of the associated hyperelliptic curve. Since we have computed that  $\Delta \equiv 2(T_2 - T_1)$ , we can recover the genus

of the hyperelliptic curve from the homology class of the corresponding rational curve via the Hurwitz formula,

$$2g - 2 = \beta \cdot \Delta = 2(b - a).$$

We can also recover the degree of the hyperelliptic curve by intersecting the rational curve with  $T_2$ .

We conclude that a generic hyperelliptic plane curve of degree  $d$  and genus  $g$  is represented by a rational curve in  $H$  of type  $(d - g + 1, d)$ . To give a more precise statement, we restrict our attention to well-behaved curves. Inside  $M_{0,0}(H, \beta)$  we look at the open substack  $M_{0,0}^{\text{tr}}(H, \beta)$  parametrizing maps from irreducible rational curves which intersect  $\Delta$  transversally. Correspondingly, in  $H_g(\mathbf{P}^2, d)$  we look at the open subset  $H_g^{\text{tr}}(\mathbf{P}^2, d)$  which parametrizes maps from smooth hyperelliptic curves such that no two points which are hyperelliptically conjugate map to the same point, and such that the differential is injective at the branch points of the hyperelliptic map. Note that this condition is equivalent to requiring that the induced map from the hyperelliptic curve to  $\mathbf{P}^1 \times \mathbf{P}^2$  is an embedding.

**Theorem 2.4.** *There is a canonical isomorphism*

$$H_g^{\text{tr}}(\mathbf{P}^2, d) \cong M_{0,0}^{\text{tr}}(H, (d - g - 1, d)).$$

*Proof.* This follows from a relative version of the construction we gave above. Over  $H_g^{\text{tr}}(\mathbf{P}^2, d)$  we have a smooth family of hyperelliptic curves,  $\mathcal{C}$ . By considering the morphism induced by the relative canonical bundle of this family, we get a two to one map from  $\mathcal{C}$  to a smooth family of rational curves,  $\mathcal{R}$ .  $\mathcal{C}$  also comes with a map to  $\mathbf{P}^2$ , and the defining property of  $H_g^{\text{tr}}(\mathbf{P}^2, d)$  is that the induced map from  $\mathcal{C}$  to  $\mathcal{R} \times \mathbf{P}^2$  is an embedding. Thus, we have produced over  $\mathcal{R}$  a flat subscheme of  $\mathcal{R} \times \mathbf{P}^2$ . By the universal property of  $H$ , we then get a morphism from  $\mathcal{R}$  to  $H$ . Now we have produced a diagram

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & H \\ \downarrow & & \\ H_g^{\text{tr}}(\mathbf{P}^2, d) & & \end{array}$$

which induces the desired morphism from the base to the space of genus 0 maps to  $H$ .

Conversely, over  $M_{0,0}^{\text{tr}}(H, (d - g - 1, d))$  we have a family of rational curves with a map to  $H$ . Pulling back the universal subscheme gives us a double cover of this family which maps to  $\mathbf{P}^2$ . The transversality condition means that this cover is branched over an étale multisection

of the family, which ensures that it is a smooth family of hyperelliptic curves. This gives us the inverse morphism.

Equivalently, applying the above arguments to families over an arbitrary base identifies the functors represented by these two moduli spaces.  $\square$

**2.4. Curves in the Hilbert Scheme.** At this point, we can be more explicit about our strategy to count hyperelliptic curves. By what we have said so far, we can see that the number of degree  $d$  genus  $g$  hyperelliptic plane curves passing through  $3d + 1$  general points is the same as the number of irreducible rational curves of type  $(d - g - 1, d)$  in  $H$ , transverse to  $\Delta$ , meeting  $3d + 1$  general translates of  $\Gamma(p)$ . Hence, we should expect to find a relationship between our enumerative problem and the Gromov-Witten invariant  $I_{(d-g-1,d)}(T_4^{3d+1})$ . What we need to do to make this relationship precise is study the structure of stable maps to  $H$  meeting these cycles in order to account for the contributions of curves which intersect  $H$  badly. In addition, we would like to ensure that the solutions to the enumerative problem are all counted with multiplicity one.

To study these questions, we need to consider the natural action of  $\mathbf{PGL}(3)$  on  $H$ . Although  $H$  is not a homogeneous space, it is nearly homogeneous, in that this action has only 2 orbits – the dense orbit parametrizing pairs of distinct points, and the diagonal. As in the case of homogeneous spaces, one can exploit this group action in two ways: to get transversality results, and to control the dimensions and smoothness of the moduli spaces.

The transversality results will follow from an easy lemma. We say  $X$  is an *almost homogeneous space* for a group  $G$ , if  $X$  is equipped with a  $G$  action which has only finitely many orbits. These orbits will then give a stratification of  $X$ . In this context there is an analogue of the Kleiman-Bertini Theorem for homogeneous spaces.

**Lemma 2.5.** *Let  $X$  be a smooth  $G$ -almost homogeneous space and  $Y$  a smooth scheme with a morphism  $f : Y \rightarrow X$ . Take  $\Gamma$  to be a smooth cycle on  $X$  which intersects the orbit stratification properly. Then, for a general  $g$  in  $G$ ,  $f^{-1}(g\Gamma)$  will be pure dimensional of the expected dimension, and the (possibly empty) open subset  $f^{-1}(\Gamma_{\text{reg}})$  will be smooth, where  $\Gamma_{\text{reg}}$  is the subset of  $\Gamma$  on which the intersection with the orbit stratification is transverse.*

*Proof.* We know that it is impossible for any component of this fiber product to have less than the expected dimension. To see that no component can have more we can just apply Kleiman-Bertini to each orbit.

Choose an orbit  $O$  of the  $G$  action. Then applying Kleiman-Bertini to the map  $f : f^{-1}(O) \rightarrow O$  tells us that for a general translate, this part of the preimage will have at most the expected dimension. Here we need the properness of the intersection to know that the codimension of  $\Gamma \cap O$  in  $O$  is at least as large as the codimension of  $\Gamma$  in  $X$ . Since there are finitely many orbits, this gives the result.

The proof of the smoothness proceeds in a similar fashion. On each orbit we can choose a general translate that ensures that the transversality condition is obtained for the restricted map. Since there are only finitely many orbits, we can choose a translate which is general with respect to each. Then we know that for each point of  $y$  in  $f^{-1}(\Gamma)$ , we have

$$D_f(T_y Y) + T_{f(y)}(\Gamma \cap O) = TO$$

where  $O$  is the orbit containing  $f(y)$ . However, on  $\Gamma_{\text{reg}}$  we know in addition that  $T_{f(y)}\Gamma + T_{f(y)}O = T_{f(y)}X$  which forces exactly the transversality condition we want.  $\square$

Now we turn our attention to understanding how the group action can be used to control the local structure of the map space. The expected dimension of the space of unpointed maps of class  $\beta$  to a space  $X$  is given by the formula

$$d_X(\beta) = -K_H \cdot \beta + \dim(X) - 3.$$

By our calculation of the canonical class of  $H$ , we conclude that the expected dimension  $d_H(\beta)$  of the moduli spaces  $\overline{M}_{0,n}(H, \beta)$  is

$$d_H = 3b + 1.$$

Since  $H$  is not a convex space, the moduli spaces  $\overline{M}_{0,n}(H, \beta)$  do not in general have this expected dimension. The almost homogeneity of  $H$  gives us unusually good control here, though. The first observation is that if we restrict our attention to curves which intersect  $\Delta$  properly, the map spaces are as nice as we could possibly hope.

**Theorem 2.6.** *If  $f : C \rightarrow H$  is a stable map of genus 0 such that no component of  $C$  is mapped entirely into  $\Delta$ , then at the moduli point  $[f]$ ,  $\overline{M}_{0,n}(H, \beta)$  is smooth of the expected dimension.*

*Proof.* Since the action of  $\mathbf{PGL}(3)$  is transitive outside of  $\Delta$ , we can conclude that the tangent bundle,  $TH$  is generated by global sections outside of  $\Delta$ . Hence, if we have a map

$$f : C \rightarrow H$$

such that no component of  $C$  is mapped entirely into the diagonal, then the pullback,  $f^*(TH)$ , is generically generated by global sections. On

a prestable curve of genus 0, this implies that the higher cohomology vanishes. As this forces the obstruction bundle to be trivial, the result follows immediately.  $\square$

Thus, the only possible source of excess dimension in the map space comes from curves which have components mapping into the diagonal. Furthermore,  $\Delta$  itself is a homogeneous space, and therefore convex, so we can calculate exactly the dimension of the space of rational curves contained in  $\Delta$ . By adjunction we have  $c_1(T\Delta) = 2T_1 + T_2$ , so the dimension of the space of rational curves contained in  $\Delta$  is given by

$$d_\Delta = 2a + b.$$

Whenever  $d_\Delta(\beta) > d_H(\beta)$  we should expect to see moduli spaces of excess dimension. We can see from our formulas that  $d_\Delta(a, b)$  will be greater than  $d_H(a, b)$  exactly when  $a > b$ . Since  $\Delta \equiv 2(T_2 - T_1)$  this condition can be rephrased as  $\beta \cdot \Delta < 0$ . In other words the excess dimension always comes from components that are trapped in the diagonal.

It is worth mentioning that this phenomenon is not special to the variety  $H$ . Any time we have a divisor  $D$  in a variety  $X$  we can write

$$c_1(TX) = c_1(TD) + D,$$

so intersection with  $D$  essentially measures the difference in the expected dimension of curves in  $D$  and curves in  $X$ .

This looks promising, since we are obviously not interested in curves which have negative intersection with  $\Delta$ , as they cannot possibly correspond to the hyperelliptic plane curves that we ultimately want to count. Unfortunately, it is not the case that if a curve class  $\beta$  has positive intersection with the diagonal then all components of  $\overline{M}_{0,n}(H, \beta)$  have the expected dimension. For example, if we write  $\beta$  as a sum of two classes

$$\beta = \beta_1 + \beta_2$$

with  $\beta_2$  representing the class of a curve contained in  $\Delta$  such that  $\beta_2 \cdot \Delta < -2$  and  $\beta_1 \cdot \Delta > 0$  then there is a component of  $\overline{M}_{0,n}(H, \beta)$  whose general element consists of a map

$$f : C \rightarrow H$$

where  $C = C_1 \cup C_2$  is a union of two rational curves meeting in a node with  $f_*[C_1] = \beta_1$  and  $f_*[C_2] = \beta_2$ . The dimension of such a component will be

$$d_H(\beta_1) + d_\Delta(\beta_2) - 2 = 3b_1 + 1 + 2a_2 + b_2 - 2 = 3b - 1 - \Delta \cdot \beta_2$$

which is strictly greater than  $d_H(\beta)$ .

In general, there will be components of the moduli space whose general element corresponds to a highly reducible curve, and many of these components will have dimension as large or larger than the expected dimension. Any of these components could conceivably give us undesired contributions to the Gromov-Witten invariants. However, if we restrict our attention to the invariants arising from incidence conditions in the plane, almost all of these contributions are zero.

**Theorem 2.7.** *Fix a class  $\beta = (a, b)$  in  $A_1 H$  with  $b > 0$  and  $3b + 1$  general points  $p_1, \dots, p_{3b+1}$  in  $\mathbf{P}^2$ .*

- (i) *There exist at most finitely many irreducible curves of class  $\beta$  which are incident to  $\Gamma(p_i)$  for all  $i$ .*
- (ii) *All such curves will intersect  $\Delta$  transversally in points disjoint from the  $\Gamma(p_i)$ .*
- (iii) *Given an arbitrary stable map in class  $\beta$  which is incident to all the cycles, it contains a unique irreducible component which is not contained in  $\Delta$ , and that component is of class  $(a_0, b)$  for some  $a_0 \leq a$ .*

Before giving the proof of this statement, we elaborate on its consequences for stable maps to  $H$ . We immediately see that aside from the distinguished component described above, all other components are of type  $(a', 0)$ . As these curves have negative intersection with  $\Delta$ , they must be contained in the diagonal. If we think of  $\Delta$  as a  $\mathbf{P}^1$  bundle over  $\mathbf{P}^2$  via the map taking a double point to its support, (different from the projection map used earlier) this is just  $a'$  times the class of a fiber. So we can see that these curves are all just multiple covers of these fibers. The fibers are just the curves described in the first section that consist of all the schemes supported at a fixed point of the plane.

For this reason, adding a component of type  $(a, 0)$  to a stable map can never cause it to be incident to any extra points in the plane. If the  $(a, 0)$  component meets  $\Gamma(p)$  it must in fact be contained in  $\Gamma(p)$ , forcing another component of the curve to meet the cycle. Also, since different fibers are disjoint, all the fibers must be incident to the distinguished central component. The source curve looks like a comb, with the  $(a_0, b)$  component as the handle, and the  $(a', 0)$  components as teeth.

Finally, because each of the  $(a', 0)$  components is forced to pass through one of the finitely many points of intersection of the distinguished component with the diagonal, there are only finitely many candidates for the  $(1, 0)$  curves which are multiply covered. In other words, there are only finitely many potential image curves for stable maps incident to all of the cycles.

*Proof.* We will proceed by induction on the number of components of  $C$ . First we consider the case where  $C$  is irreducible. If  $C$  is not contained in  $\Delta$  we know that it moves in the expected dimension, so (i) and (ii) follow immediately from the general position lemma (since the  $\Gamma(p_i)$  intersect  $\Delta$  properly, and since a general irreducible curve will intersect  $\Delta$  transversally.)

If, on the other hand,  $C$  were contained in  $\Delta$ , then  $C$  would give rise to a non-reduced subscheme of  $\mathbf{P}^2$  which was supported on a rational curve of degree  $\frac{b}{2}$ .  $C$  would meet  $\Gamma(p_i)$  if and only if this subscheme met  $p_i$ , but a rational curve of degree  $\frac{b}{2}$  can meet at most  $\frac{3b}{2} - 1$  general points (if  $b > 0$ ).

Now, suppose we have a reducible curve which meets the  $\Gamma(p_i)$ . From what we have already said, it follows that it cannot be contained in  $\Delta$ . We also know that it must have at least one component contained in  $\Delta$ , since this is the only possible source of excess dimension. Hence, we can write our curve as a union  $C_0 \cup C_1$  with the point of intersection  $C_0 \cap C_1$  contained in the diagonal. By the inductive hypothesis, each  $C_i$  can meet at most  $3b_i + 1$  of the cycles. Without loss of generality, we can assume that  $C_0$  meets  $3b_0 + 1$  and  $C_1$  meets  $3b_1$ . Again by induction, we know that there exist only finitely many image curves for  $C_0$ , and the only components of  $C_0$  which are contained in the diagonal are of type  $(a', 0)$ . This means that all potential points of  $C_0 \cap \Delta$  can be contained in a union of finitely many cycles of the form  $\Gamma(q_i)$ . This would then imply that  $C_1$  would have to meet at least one of the  $\Gamma(q_i) \cap \Delta$  and meet an additional  $3b_0$  of the cycles. But this is just asking it to meet  $3b_0 + 1$  general cycles, and to have one of the points of intersection in  $\Delta$  which is impossible.  $\square$

### 3. VIRTUAL CONTRIBUTIONS

**3.1. Smoothness.** Theorem 2.7 gives us a very sharp picture of the locus

$$A = \pi(\rho_1^{-1}\Gamma_1 \cap \cdots \cap \rho_n^{-1}\Gamma_n).$$

Since the only moduli in the choice of a stable map meeting all of the cycles is the choice of multiple cover of the  $(1, 0)$  curves,  $A$  is a union of finitely many components, each of which set theoretically decomposes as a product

$$M(a_1) \times M(a_2) \times \cdots \times M(a_n).$$

Here  $M(a)$  is the space parametrizing the data of an  $a$ -sheeted cover of  $\mathbf{P}^1$  with a choice of point mapping to 0 (the point of attachment to the distinguished component.) This space is just a fiber of the evaluation map  $\rho_1 : \overline{M}_{0,1}(\mathbf{P}^1, a) \rightarrow \mathbf{P}^1$ .

Since we really want to count just the irreducible curves, we need to be able to determine the contribution that each of these components makes to the Gromov-Witten invariant. To do this, we first observe that we are in the relatively simple situation described in Theorem 1.3. Namely, we just need to intersect the virtual class with a complete subvariety which is completely contained in the smooth locus.

We want to prove that in a neighborhood of one of these comb curves, the moduli space of maps is smooth. The key step in proving this is to show that there are no first order deformations of such a map which resolve the nodes. Once this is proven, the smoothness of the moduli space follows from the smoothness of the moduli space parametrizing the handle of the comb, and the smoothness of the spaces parametrizing the teeth. It also follows that the locus  $A$  defined above with its natural scheme structure is reduced, by our transversality lemma.

We write  $C = C_0 \cup C_1 \cup \dots \cup C_n$  where  $C_0$  is the handle, and we assume we are given a map  $f : C \rightarrow H$  which takes  $C_0$  to a curve intersecting  $\Delta$  transversally, and the other  $C_i$  to multiple covers of  $(1, 0)$  curves. We write  $p_i$  to denote the node where  $C_i$  meets  $C_0$ . Intuitively, the reason there can be no smoothings of such a map is that if we look at a small neighborhood of one of the components in  $\Delta$ , we see that  $C_0$  meets  $\Delta$  only once, while  $C_i \cdot \Delta \leq -2$ . A small neighborhood of  $p_i$  on the smoothed curve would then have to contribute  $-1$  to the intersection with  $\Delta$ . This immediately precludes any actual smoothings, but we need to verify that it also rules out the existence of a first order smoothing.

Suppose we have a flat family,  $\pi : \tilde{C} \rightarrow S$ , over the double point,  $S = \text{Spec } \mathbb{C}[t]/t^2$ , and a morphism  $\tilde{f} : \tilde{C} \rightarrow H$  such that the restriction to the reduced point is as above. To say that this family smoothes the node at  $p_i$  means that in a small neighborhood of  $p_i$ ,  $\tilde{C}$  looks like a first-order neighborhood of a planar node. This scheme can naturally be written as a union of a first order neighborhood of each of the two components, so  $\tilde{C}$  can naturally be written as the union of  $\tilde{C}_j$ . By embedding the first order deformation in a global deformation, it is easy to see that  $\tilde{C}_i$  is isomorphic to a first order neighborhood of  $\mathbf{P}^1$  in the total space of  $\mathcal{O}(-1)$ . Then, pulling back  $\mathcal{O}(\Delta)$  along with its tautological section gives us a line bundle  $L$  on  $\tilde{C}$  whose degree on  $C_i$  is less than  $-1$ , but with a global section, whose restriction to  $C_0$  vanishes only to order 1 at  $p_i$ . The intersection of  $\tilde{C}_i$  with  $C_0$  is a double point, so the section cannot be zero on  $\tilde{C}_i$ .

We are now reduced to proving that there do not exist any line bundles on  $\tilde{C}_i$  whose degree on  $C_i$  is less than  $-1$  and have a non-zero section. This will follow from two lemmas.

**Lemma 3.1.** *All line bundles on  $\tilde{C}_i$  are pulled back from the projection map to  $\mathbf{P}^1$ .*

*Proof.* Twisting up by a high power of  $\mathcal{O}(1)$  we can assume that our line bundle has a section whose restriction to  $C_i$  has only simple zeroes. It is clear that any such Cartier divisor on  $\tilde{C}_i$  extends to a divisor on the total space of  $\mathcal{O}(-1)$ . Now the result follows from the familiar fact that the Picard group of the total space of a vector bundle is equal to the Picard group of the base.  $\square$

We now know that the restriction of  $L$  to  $\tilde{C}_i$  is of the form  $\mathcal{O}(-d)$  for some  $d$  greater than or equal to 2. All that remains is to show that these bundles have no sections.

**Lemma 3.2.** *Let  $X$  be a scheme, with  $E$  and  $F$  vector bundles over  $X$ . Let  $\tilde{X}$  be the first order neighborhood of  $X$  in  $E$ , with  $\pi : \tilde{X} \rightarrow X$  the natural projection map. There is a canonical isomorphism*

$$H^0(\tilde{X}, \pi^* F) \cong H^0(X, F) \oplus \text{Hom}(E, F).$$

*Proof.* Pullback and restriction of sections give us the sequence

$$H^0(X, F) \rightarrow H^0(\tilde{X}, \pi^* F) \rightarrow H^0(X, F)$$

which shows that  $H^0(X, F)$  is a direct summand. We need to show that the kernel of the restriction map is  $\text{Hom}(E, F)$ . Given an element of  $\text{Hom}(E, F)$  we interpret it as a morphism between the total spaces of the bundles. Restricting to  $\tilde{X}$  gives us a section which vanishes on  $X$ . The inverse of this map is given by interpreting an element of  $H^0(\tilde{X}, \pi^* F)$  as a morphism over  $X$  from  $\tilde{X}$  to the total space of  $F$ . The differential of this map gives a morphism of relative tangent bundles from  $T_{\tilde{X}/X}$  to  $T_{F/X}$ , which are naturally identified with  $E$  and  $F$  respectively.  $\square$

Applying this lemma with  $X = \mathbf{P}^1$ ,  $E = \mathcal{O}(-1)$ , and  $F = \mathcal{O}(-d)$  completes the proof of the smoothness of the moduli space.

**3.2. Tangent-Obstruction Sequence.** In order to apply Theorem 1.3 to compute the contributions from the positive dimensional components, we need to understand the obstruction bundle  $\mathcal{E}$  on these loci.

Our only tool to study this bundle is the tangent obstruction sequence:

$$0 \rightarrow H^0(TC) \rightarrow H^0(f^*TH) \rightarrow TM \xrightarrow{\phi} \text{Ext}^1(\Omega_C, \mathcal{O}_C) \rightarrow H^1(f^*TH) \rightarrow \mathcal{E} \rightarrow 0.$$

From what we have already done, we know that the cokernel of  $\phi$  has dimension at least  $n$ , since none of the nodes can be smoothed. In fact, the cokernel has dimension exactly  $n$ . That is, the family of deformations of the map surjects onto the topologically trivial deformations of the curve. This is equivalent to saying that the deformations of the irreducible curve realize all possible deformations of configurations of  $n$  points in  $\mathbf{P}^1$  via intersection with  $\Delta$ . Translating this back into the language of hyperelliptic plane curves, it just means that the map from the moduli space of hyperelliptic maps to the space of hyperelliptic curves is smooth which was proven in Section 2.2.

This means that the obstruction bundle fits into an exact sequence:

$$(3.3) \quad 0 \rightarrow \bigoplus_1^n L_i \rightarrow H^1(f^*TH) \rightarrow \mathcal{E} \rightarrow 0.$$

Here  $L_i$  is the line bundle corresponding to the deformation which resolves the  $i^{\text{th}}$  node.

To analyze the middle term of this sequence, we look at the normalization sequence:

$$(3.4) \quad \begin{aligned} 0 \rightarrow & H^0(C, f^*TH) \rightarrow \bigoplus_0^n H^0(C_i, f_i^*TH) \rightarrow \\ & \rightarrow \bigoplus_1^n TH_{p_i} \rightarrow H^1(C, f^*TH) \rightarrow \bigoplus_0^n H^1(C_i, f_i^*TH) \rightarrow 0. \end{aligned}$$

We know that  $H^1(C_0, f_0^*TH) = 0$  by almost homogeneity, and since  $H^1(f_i^*T\Delta) = 0$  for the same reason, we can conclude that  $H^1(C_i, f_i^*TH) = H^1(C_i, f_i^*N_{\Delta/H})$ . As  $N$  has degree  $-2a_i$  on  $C_i$ , the rank of the last bundle in sequence 3.4 is  $\sum_1^n (2a_i - 1)$ .

However, since we know by smoothness that the rank of  $\mathcal{E}$  is  $\sum_1^n (2a_i - 2)$ , sequence 3.3 forces

$$H^1(C, f^*TH) \cong \bigoplus_1^n H^1(C_i, f_i^*N).$$

Hence, we can rewrite the end of the tangent obstruction sequence as

$$0 \rightarrow \bigoplus L_i \rightarrow \bigoplus H^1(C_i, f_i^*N) \rightarrow \mathcal{E} \rightarrow 0.$$

In fact, this sequence naturally splits as a direct sum of  $n$  exact sequences

$$0 \rightarrow L_i \rightarrow H^1(C_i, f_i^*N) \rightarrow \mathcal{E}_{a_i} \rightarrow 0$$

where  $\mathcal{E}_a$  is a vector bundle of rank  $2a - 2$  on  $M(a)$ . This bundle is the one that arises when  $n = 1$ . We won't prove this splitting here, since we are only interested in the top Chern class of  $\mathcal{E}$ , and we can already

see that the Chern classes of  $\mathcal{E}$  must be equal to the Chern classes of  $\bigoplus \mathcal{E}_{a_i}$  since these bundles fit into the same exact sequence.

**3.3. Evaluation of the Euler Class.** The results of the previous section reduce our problem to evaluating the top Chern class of  $\mathcal{E}_a$  which is a vector bundle on  $M(a)$ . We will see that this Chern class vanishes, so it would be nice to exhibit either a trivial subbundle or trivial quotient bundle of  $\mathcal{E}_a$ . We have been unable to find such a bundle. Instead we use a trick. We will realize  $M(a)$  as the moduli space of maps to another variety, in such a way that the obstruction bundle will again be  $\mathcal{E}_a$ . The number  $c_{\text{top}}(\mathcal{E}_a)$  will then be identified as a Gromov-Witten invariant of this variety, and we will be able to evaluate it using the deformation invariance property.

We consider the variety  $X$  obtained by blowing up  $\mathbf{P}^2$  at a point, and then blowing up a point on the exceptional divisor. This gives us a surface with two exceptional divisors  $A$  and  $B$  meeting in a node. We take  $A$  to be the  $-1$  curve and  $B$  to be the  $-2$  curve and set  $\beta_a = A + aB$ . All representative stable maps in class  $\beta_a$  consist of a map from a reducible curve with one component mapping isomorphically onto  $A$  and the rest forming an  $a$  sheeted cover of  $B$ . The same arguments as before show that the moduli space of maps,  $\overline{M}_{0,0}(X, \beta_a)$  is smooth and isomorphic to  $M(a)$ . The expected dimension of these moduli spaces is 0, independent of  $a$ . So for each  $a$  there is a zero point Gromov-Witten invariant,  $I_{\beta_a}^X$ , which is just the degree of the virtual fundamental class. As the whole moduli space is smooth, again we can realize this class as the top Chern class of a vector bundle which sits in the exact sequence

$$0 \rightarrow L \rightarrow H^1(f^*TX) \rightarrow \mathcal{E}_a \rightarrow 0$$

where  $L$  is the line bundle corresponding to smoothing the unique node lying on the component of the curve which maps to  $A$ . Finally, since the normal bundle to  $B$  has degree  $-2$ , we can realize the middle term as  $H^1(C, f^*\mathcal{O}(-2))$ . This confirms that we are indeed looking at the same vector bundles as arise in determining the Gromov-Witten invariants of  $H$  (at least in K-theory).

Now we are finally in a position to prove the vanishing result we are after.

**Proposition 3.5.** *For all  $a \geq 2$ ,  $c_{\text{top}}(\mathcal{E}_a) = 0$ .*

*Proof.* We have already seen that this Chern class is equal to  $I_{\beta_a}^X$ . We consider a one parameter family of smooth varieties specializing to  $X$ . We construct it by starting with  $\mathbb{A}^1 \times \mathbf{P}^2$  and first blowing up  $\mathbb{A}^1 \times [1, 0, 0]$ , and then blowing up the proper transform of the locus  $\{(t, [1, 0, t])\}$ . If we denote by  $X_t$  the fiber of the projection to  $\mathbb{A}^1$ ,  $X_t$  is

the blow-up of  $\mathbf{P}^2$  at  $[1, 0, 0]$  and  $[1, 0, t]$ , and  $X_0 \cong X$ . By deformation invariance, we can compute our invariant on  $X_1$  instead. We just need to know which homology class  $\beta_a$  corresponds to. If we label the two exceptional divisors in  $X_t$  as  $D_0$  and  $D_t$  then  $D_0$  specializes to  $A + B$  and  $D_t$  specializes to  $B$ . So the class corresponding to  $\beta_a = A + aB$  is  $D_t + a(D_0 - D_t)$  which cannot be represented by any effective curves if  $a \geq 2$ . This forces the vanishing of Gromov-Witten invariants in this class.  $\square$

With this result, we can conclude the main theorem about the relationship between Gromov-Witten invariants of  $H$  and enumerative geometry of hyperelliptic plane curves. We set  $E(d, g)$  to be the number of hyperelliptic curves passing through  $3d + 1$  general points.

**Theorem 3.6.** *The enumerative numbers  $E(d, g)$  satisfy the equation*

$$I_{(d-g-1, d)}(T_4^{3d+1}) = \sum_{h \geq g} \binom{2h+2}{h-g} E(d, h).$$

*Proof.* We know that there is a zero dimensional component of the moduli space corresponding to curves of the following type. Take a curve of type  $(a', b)$  incident to all of the  $\Gamma(p_i)$  and attach to it  $a - a'$  rational curves each mapping isomorphically onto a  $(1, 0)$  curve. The number of such maps is equal to the number of irreducible  $(a', b)$  curves through the cycles times the number of choices for attachment points of the  $(1, 0)$  curves. The  $(a', b)$  curve will meet the diagonal in  $2b - 2a'$  points, of which we choose  $a - a'$ . Also, by part (ii) of Theorem 2.7 and our general position result, it follows that these zero dimensional components are reduced, and so count with multiplicity one. The formula then follows immediately from the relationship between  $(a, b)$  and  $(d, g)$ , since we have established that the only positive dimensional components come from looking at higher degree multiple covers of the  $(1, 0)$  curves and these make no contribution to the Gromov-Witten invariant.  $\square$

**3.4. Genus 0 and 1.** The  $E(d, g)$  will all be 0 for curves of genus 0 or 1, since it is well known that curves of genus 0 can meet at most  $3d - 1$  points, and elliptic curves can meet at most  $3d$  points. From our point of view, this is because these curves have extra  $g_2^1$ s. It is still true that the rational curves in  $H$  corresponding to these curves move in a  $3d + 1$  dimensional family, but the correspondence between a hyperelliptic curve in  $\mathbf{P}^2$  and a rational curve in  $H$  actually chooses a hyperelliptic curve with a choice of hyperelliptic involution. On curves of genus greater than 1, this is no choice at all, since such a curve

can have at most one hyperelliptic involution. Elliptic curves have a 1-parameter family of  $g_2^1$ s, and rational curves have a 2-parameter family. We can still use the Gromov-Witten invariants of  $H$  to compute the genus 0 and 1 Severi degrees however. This is done by looking at a slightly larger set of Gromov-Witten invariants and by considering the corresponding wider class of enumerative problems.

We will solve the following enumerative problem: given  $k$  general points,  $p_1, \dots, p_k$ , and  $l$  general pairs of points,  $q_1, r_1, \dots, q_l, r_l$ , with  $k + 3l = 3d + 1$ , how many hyperelliptic curves of genus  $g$  and degree  $d$  pass through all the points, and satisfy the additional condition that for some choice of hyperelliptic involution,  $q_i$  is hyperelliptically conjugate to  $r_i$  for all  $i$ . If we set  $E^l(d, g)$  to be the solution to this problem, then we get the following result.

**Theorem 3.7.**

$$I_{(d-g-1,d)}(T_8^l \cdot T_4^{3(d-l)+1}) = \sum_{h \geq g} \binom{2h+2}{h-g} E^l(d, h)$$

*Proof.* The connection to Gromov-Witten invariants is that a hyperelliptic curve will satisfy the condition that it meets  $q$  and  $r$  and has the corresponding points as hyperelliptic conjugates if and only if the associated rational curve meets the point in  $H$  parametrizing the subscheme  $\{q, r\}$ . So we can see that the solution to our enumerative problem is exactly the number of irreducible curves in  $H$  of the appropriate class meeting  $l$  general points, and  $k$  general translates of  $\Gamma(p)$ . Now we can apply the same arguments as were used in the proof of Theorem 3.6 to conclude this result. The only place in our proof where we needed to refer to the classes that were being intersected was in Theorem 2.7 where we showed that almost no reducible curves could contribute. As we choose our representative of the point class to be a point outside of  $\Delta$ , no curves moving in excess dimension can satisfy this condition, and the exact same argument will hold for the invariants involving this class.  $\square$

As any pair of points on an elliptic curve are conjugate under a unique hyperelliptic involution, and any 2 pairs of points determine a unique hyperelliptic map on  $\mathbf{P}^1$ , we can conclude that the genus 0 and 1 Severi degrees are given by  $E^2(d, 0)$  and  $E^1(d, 1)$  respectively. Theorem 3.7 then gives us a means of computing these degrees. In genus 0, of course, one can compute these Severi degrees much more readily by working directly with the genus 0 Gromov-Witten invariants of the plane. Computing them this way does provide a useful check on our calculations, though.

**3.5. Hirzebruch Surfaces.** We remark that the exact same calculations apply to the problem of finding the genus 0 Severi degrees of the Hirzebruch surface  $\mathbb{F}_2$ , recovering a result of Abramovich and Bertram, [AB] (see [V] who computes Severi degrees of all genera of ruled surfaces, and includes a discussion of the result mentioned.)

We just sketch how this application can be carried out. The moduli spaces of rational maps to  $\mathbb{F}_2$  all have the expected dimension, except for curves which involve a multiple cover of the  $-2$  curve. Just as in the double blow-up of  $\mathbf{P}^2$ , the obstruction theory for curves with components mapping onto the exceptional divisor is identical to that of curves in  $H$  with components mapping multiply onto  $(1, 0)$  curves. One can conclude that exactly the same formula as in Theorem 3.6 relates the genus 0 Severi degrees to the Gromov-Witten invariants on  $\mathbb{F}_2$ . In this case, one can say even more, since it follows by deformation invariance that the Gromov-Witten invariants of  $\mathbb{F}_2$  are identical to those for  $\mathbb{F}_0$ . As  $\mathbb{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$  is a homogeneous space, we conclude that the Gromov-Witten invariants of  $\mathbb{F}_2$  are equal to appropriate genus 0 Severi degrees on  $\mathbb{F}_0$ , so the formula can be interpreted as relating the genus 0 Severi degrees of the two varieties. Set  $N_{\mathbb{F}_i}(D)$  to be the genus 0 Severi degree of  $\mathbb{F}_i$  for the linear series  $D$ . On  $\mathbb{F}_i$ , take  $F$  to be the class of a fiber,  $S$  to be the class of a section with self intersection  $i$ , and  $E$  to be the class of a section with self intersection  $-i$ . The result is the formula

$$N_{\mathbb{F}_0}(aS + (b + a)F) = \sum_{i=0}^{a-1} N_{\mathbb{F}_2}(aS + bF - iE).$$

Abramovich and Bertram obtain this result by studying the deformation which specializes  $\mathbb{F}_0$  to  $\mathbb{F}_2$ . This is of course closely analogous to the deformation which we use in the proof of Theorem 3.5. In fact, it was the similarity between their result and Theorem 3.6 which suggested to us the possibility of a proof via surface geometry. (Our original proof was along different lines.)

#### 4. CALCULATION OF GROMOV-WITTEN INVARIANTS

**4.1. Two Point Invariants.** To actually compute the enumerative numbers we are after, we need a method to calculate the Gromov-Witten invariants of  $H$ . The main tool here will be the First Reconstruction Theorem of [KM]. This result (mentioned in Section 1) gives an explicit algorithm by which all Gromov-Witten invariants of  $H$  can be computed in terms of just the 2 point numbers. We can apply this theorem because we have already seen that the cohomology ring of  $H$  is

generated by divisors. This immediately reduces our problem to finding all numbers of the form  $I_{(a,b)}(\gamma_1 \cdot \gamma_2)$ . Since  $H$  is four dimensional, each  $\gamma_i$  can impose at most 3 conditions on curves, so together, 2 classes can impose at most 6 conditions. However, if  $b > 1$  the expected dimension of curves of type  $(a, b)$  is at least 7, so 2 point numbers exist only for curves of types  $(a, 0)$  and  $(a, 1)$ .

We start by looking at  $(a, 0)$  curves. We have already observed that all curves of this type are  $a$ -sheeted covers of curves of type  $(1, 0)$ . The expected dimension of the space of  $(a, 0)$  curves is 1, independent of  $a$ . So the only invariants here are of the form  $I_{(a,0)}(\gamma)$  where  $\gamma$  is an element of  $A^2 H$ .

We first compute the invariant on the two geometric codimension two loci  $T_4$  and  $S_5$ .

**Lemma 4.1.** *For all  $a$ ,  $I_{(a,0)}(T_4) = I_{(a,0)}(S_5) = 0$ .*

*Proof.* In each case, the vanishing of the invariant is because the class actually imposes 2 conditions on  $(a, 0)$  curves. Consider the  $a = 1$  case. We can see that for a  $(1, 0)$  curve to be incident to  $\Gamma(p)$ , we have to take the curve to be the curve of nonreduced schemes supported at  $p$ . This condition is obviously codimension 2 in the space of all  $(1, 0)$  curves. We were only expecting a codimension 1 condition, and as a result we see that  $I_{(1,0)}(T_4) = 0$  since  $\Gamma(p)$  represents  $T_4$ . Similarly for an  $(a, 0)$  curve to meet  $\Gamma(p)$ , the  $(1, 0)$  curve which it covers must be the one supported at  $p$ . This is again a codimension 2 condition.

For the  $S_5$  invariant, we use the representation of  $S_5$  as the cycle of subschemes incident to each of 2 lines. It is easy to see that a  $(1, 0)$  curve can meet this cycle only if it is the curve consisting of subschemes supported at the intersection of the two lines. Now the same argument that we used for  $T_4$  gives us the result.  $\square$

This leaves  $I_{(a,0)}(T_3)$ . We will compute this directly for  $a = 1$ . Under the natural identification of  $\overline{M}_{0,0}(H, (1, 0))$  with  $\mathbf{P}^2$ , the locus of curves incident to a general representative of  $T_3$  is a line. It follows that  $I_{(1,0)}(T_3)$  is simply equal to the degree of the virtual class.

In this case, we can directly compute the virtual class. We have that the moduli space is just  $\mathbf{P}^2$ , and the universal curve over it is the variety of complete flags in  $\mathbf{P}^2$  which maps isomorphically onto  $\Delta$ . The virtual class is then  $c_1(R^1\pi_*(f^*N_{\Delta/H}))$ . As we know that the normal bundle is strictly negative on all curves in this class, it follows that  $\pi_*f^*N = 0$ , so

$$R^1\pi_*f^*N = -\pi_!(f^*N).$$

We know what the Chern class of  $N_{\Delta/H}$  is, since we have computed its intersection with each of the curve classes. It is now straightforward to apply Grothendieck-Riemann-Roch to compute the Chern character of this bundle and read off the desired Chern class. The result is that  $I_{(1,0)}(T_3) = 3$ . We will later calculate  $I_{(a,0)}(T_3)$  for  $a > 1$  by means of the associativity relations. It is also possible to compute  $I_{(1,0)}(T_3)$  by making use of more of the associativity relations, thereby avoiding the direct virtual class calculation.

Now consider the curves of type  $(a, 1)$ . The main result we want to prove about these curves is the following:

**Theorem 4.2.** *If  $a > 2$ , then all Gromov-Witten invariants for curves of type  $(a, 1)$  vanish.*

*Proof.* We observe that such curves have negative intersection with the diagonal, but on the other hand, cannot be contained in the diagonal since 1 is odd. Hence, all such curves are reducible, and have a  $(0, 1)$  curve as one component. (Irreducible curves of type  $(1, 1)$  do not intersect the diagonal, and so cannot be a component of a connected curve whose other components are all contained in  $\Delta$ .) A general  $(0, 1)$  curve will intersect  $\Delta$  in two distinct points, so a general curve of type  $(a, 1)$  will be a union of an  $(a_1, 0)$  curve, a  $(a_2, 0)$  curve, and a  $(0, 1)$  curve where  $a_1 + a_2 = a$ .

We are free to choose a basis of cohomology such that every class in our basis can be represented by cycles which intersect  $\Delta$  properly. Hence, it suffices to show that any Gromov-Witten invariant involving such classes is 0. By choosing such cycles, we can guarantee that the preimages of general translates will intersect in a subscheme of the correct codimension, which in this case is 6. We then expect the codimension of the image of this intersection in the unpointed space to be 4. Of course, as we saw in our discussion of the  $(a, 0)$  curves, it is possible for the image of the intersection to have less than the expected dimension. In this case there is nothing to prove, though, since the invariants are forced to vanish.

We assume then that the image of the intersection of the cycles has codimension 4 in the space of unpointed maps. This fact alone is enough to determine very precisely what the structure of this locus is. We consider the map

$$g : \overline{M}_{0,0}(H, (a, 1)) \rightarrow \overline{M}_{0,0}(H, (0, 1))$$

which forgets the components contained in the diagonal. Then we observe that if a given point is in the intersection, that all points in that component of the fiber of  $g$  must also be in the intersection. This is

simply because the difference between stable maps that lie in the same fiber of  $g$  is just the choice of multiple cover of the associated  $(1, 0)$  curves. Choosing a different cover cannot affect incidence conditions. Since these fibers are already codimension 4, it follows that the intersection consists entirely of a union of finitely many components of fibers of  $g$ .

Thus, the structure of the solution sets is of the form

$$\coprod M(a_1) \times M(a_2)$$

with  $a_1 + a_2 = a$  where these are the same  $M(a)$  that occurred in the previous section. The virtual classes here are clearly the same as they were in that context, so we can conclude that any time one of the  $a_i$  is greater than 1, the contribution from that component is 0. If  $a > 2$ , then for some  $i$ ,  $a_i > 1$ , so we are done.  $\square$

To calculate the 2-point numbers for curves of type  $(a, 1)$  with  $a \leq 2$  we simply solve directly the appropriate enumerative problems, being careful to do our computations with cycles that intersect  $\Delta$  properly. In the  $a = 2$  case, we apply the same virtual calculation to disregard solutions involving double covers of a  $(1, 0)$  curve. This is elementary, but somewhat tedious. We compile the results in a table.

$I_{(a,1)}(T_i, T_j)$	$(i, j) = (3, 8)$	$(4, 8)$	$(5, 8)$	$(6, 6)$	$(6, 7)$	$(7, 7)$
a=0	0	0	1	0	0	1
1	1	2	1	1	2	-2
2	1	0	0	4	-2	1

This leaves only the numbers  $I_{(a,0)}(T_3)$ , which we will now compute via the associativity equations.

We will need to use only the equation associated to the following diagram

$$\binom{6}{3} \succ \binom{2}{1} \sim \binom{6}{3} \diagup \binom{2}{1}$$

with no cohomology classes and  $\beta = (a, 1)$ .

Most of the terms in this equation vanish immediately. Since  $T_2 \cdot (a, 0) = 0$ , the divisor axiom forces any Gromov-Witten invariant for  $(a, 0)$  curves containing a  $T_2$  to vanish. This already simplifies the recursion to

$$I_{(a,1)}(T_3, T_6, T_4+2T_3) - I_{(a,1)}(T_1, T_3, T_8) - \sum_{a_1+a_2=a} I_{(a_1,0)}(T_1, T_1, T_3) \cdot I_{(a_2,1)}(T_2, T_6, T_7) = 0.$$

Applying this in case  $a > 2$  we can use the vanishing of  $(a, 1)$  invariants and our calculation of  $I_{(a_2, 1)}(T_6, T_7)$  to reduce to

$$(a-1)^2 I_{(a-1, 0)}(T_3) = (a-2)^2 I_{(a-2, 0)}(T_3)$$

Given our computation that  $I_{(1, 0)}(T_3) = 3$ , this inductively determines that  $I_{(a, 0)}(T_3) = 3/a^2$ .

**4.2. Enumerative Results.** The results of the previous section give us complete knowledge of all 2-point numbers on  $H$ , so we can recursively determine arbitrary Gromov-Witten invariants by means of the First Reconstruction Theorem. This algorithm was implemented on Maple. We list here some of the results and their enumerative consequences.

$I(T_4^{3d+1})$	d=2	3	4	5	6	7
g=0	0	0	405	560385	1096808499	3292618732704
1	*	0	162	224910	460743174	1470159619803
2	*	*	27	37935	89898984	338090337018
3	*	*	*	135	3933549	29267016849
4	*	*	*	*	405	539160678
5	*	*	*	*	*	945
$E(d, g)$						
g=0	0	0	0	0	0	0
1	*	0	0	0	0	0
2	*	*	27	36855	58444767	122824720116
3	*	*	*	135	3929499	23875461099
4	*	*	*	*	405	539149338
5	*	*	*	*	*	945

The zeros occur in the genus 0 and 1 rows of the enumerative table because of the extra hyperelliptic  $g_2^1$ 's on rational and elliptic curves. Since every genus 2 curve is hyperelliptic,  $E(d, 2)$  is the genus 2 Severi degree in the linear series of degree  $d$  plane curves. These numbers agree with the calculations of [CH] and have now been computed again by a different method in [BP].  $E(4, 2) = 27$  is the degree of the quartic discriminant hypersurface.  $E(5, 2) = 36855$  can also be computed via the 4-nodal formula due to I. Vainsencher.

$I(T_4^{3d-2} * T_8)$	d=2	3	4	5	6	7
g=0	0	4	975	500070	510209009	936943088028
1	*	1	255	147780	172751014	358483479813
2	*	*	5	10138	21081609	61683241918
3	*	*	*	12	558749	3685184208
4	*	*	*	*	22	32184102
5	*	*	*	*	*	35
$E^1(d, g)$						
g=0	0	0	0	0	0	0
1	*	1	225	87192	57435240	60478511040
2	*	*	5	10042	16612387	33328207904
3	*	*	*	12	558529	3363345078
4	*	*	*	*	22	32183682
5	*	*	*	*	*	35

The genus 0 row still vanishes. As we mentioned in section 3.4, the row  $E^1(d, 1)$  gives the genus 1 Severi degrees of  $\mathbf{P}^2$ . These elliptic numbers agree with computations by E. Getzler in [G] (who also checked  $E_{1,6} = 57435240$  with the algorithm of L. Caporaso and J. Harris, [CH]).

$I(T_4^{3d-5} * T_8^2)$	d=2	3	4	5	6	7
g=0	1	16	1279	317408	187613888	222541278466
1	*	1	167	63228	49635964	72095337199
2	*	*	1	2536	4254399	9650092804
3	*	*	*	1	65417	402592233
4	*	*	*	*	1	1900762
5	*	*	*	*	*	1
$E^2(d, g)$						
g=0	1	12	620	87304	26312976	14616808192
1	*	1	161	48032	25417860	22151587040
2	*	*	1	2528	3731098	6495881498
3	*	*	*	1	65407	383584667
4	*	*	*	*	1	1900750
5	*	*	*	*	*	1

$E^2(d, 0)$  equals  $N_d$ , the number of degree  $d$  rational curves passing through  $3d - 1$  general points in  $\mathbf{P}^2$ . The numbers  $N_d$  agree with the computation of M. Kontsevich.

**4.3. Small Quantum Cohomology.** In this section we use our results to give an explicit presentation of the small quantum cohomology ring of  $H$ . We will see that essentially all of the work has been done

in the computations of Section 4.1. The general principle at work is that if a variety has its cohomology generated by divisors, then the small quantum cohomology can be described in terms of the two point Gromov-Witten invariants. From one point of view, this is an immediate consequence of the reconstruction theorem used earlier. However, in that context there were infinitely many invariants to be determined, and we merely had a recursive procedure to find them. Here, because we know that we just need to find a finite set of relations, the problem can be solved in closed form.

We remind the reader of the definition of the small quantum product. We introduce variables  $q_1$  and  $q_2$ , and define a multiplication on  $A^*(H)[[q_1, q_2]]$ . Given  $\gamma_1$  and  $\gamma_2$  in  $A^*(H)$  we set

$$\gamma_1 * \gamma_2 = \sum I_{(a,b)}(\gamma_1 \cdot \gamma_2 \cdot T_i) q_1^a q_2^b T_{8-i}.$$

The product is extended to the whole power series by linearity over  $\mathbb{Q}[[q_1, q_2]]$ . The associativity of this product is a consequence of the WDVV equations.

First we compute the quantum product of divisors with elements of  $A^1$  and  $A^2$ . (We can get by without computing other products because the relations in the ordinary cohomology ring are of degree 3.) The result is as follows.

$$\begin{aligned} T_1 * T_1 &= (1 - 3f)T_3 + 3fT_5 \\ T_1 * T_2 &= 2T_3 + T_4 \\ T_2 * T_2 &= T_3 + T_4 + T_5 \\ T_1 * T_3 &= 3fT_7 + q_1 q_2 + 2q_1^2 q_2 \\ T_1 * T_4 &= T_6 + 2q_1 q_2 \\ T_1 * T_5 &= 2T_6 + (1 - 3f)T_7 + q_1 q_2 \\ T_2 * T_3 &= T_6 + q_1 q_2 + q_1^2 q_2 \\ T_2 * T_4 &= T_6 + T_7 + 2q_1 q_2 \\ T_2 * T_5 &= T_6 + 2T_7 + q_2 + q_1 q_2 \end{aligned}$$

where

$$f = \frac{q_1}{1 - q_1} = q_1 + q_1^2 + q_1^3 + \dots$$

This table follows immediately from the definition of quantum product, the analogous calculation in the ordinary cohomology ring, the divisor axiom, and the two point invariants computed in Section 4.1. Note that unlike the situation with Fano manifolds, the quantum multiplication is not defined at the polynomial level, but only in terms of formal power series.

From this we can explicitly write all triple products of divisors in terms of  $T_6$  and  $T_7$ . It is then just some linear algebra to deduce the following relations.

$$T_1 * T_1 * T_1 = 9f^2 T_1 * T_2 * T_2 - (9f^2 - 2f) T_2 * T_2 * T_2 + q_1 q_2 (q_1 - 1) (1 - 18f) T_2 * T_2 * T_2 - 3(1 - 6f) T_1 * T_2 * T_2 + 6T_1 * T_1 * T_2 = q_2 (q_1^2 - 2q_1 + 1)$$

As these are deformations of the two relations defining the ordinary cohomology ring of  $H$ , it follows that they give a complete set of relations for the quantum ring. It is interesting that by clearing denominators, it is in fact possible to write these relations as *polynomials* and not just power series. However, the  $T_i$  would not be in the polynomial ring generated by the divisors, only in the ring localized at  $1 - q_1$ .

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